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Diff S^1 as coadjoint orbit in the Virasoro space of moments

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This paper presents an embedding of Diff(S^1) as a symplectic coadjoint orbit of itself and of the Virasoro group with non-zero central charge.

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1. Introduction

The so-called regular coadjoint orbits of $\mathrm{Diff}^+(S^1)$ have been investigated by many authors ([1] and [4] for example). The group itself may appear in the list of those orbits. Also, in the case of the regular orbits of the Virasoro Group (an universal central extension of $\mathrm{Diff}^+(S^1)$) one knows that, for a non-zero central charge, they are all of finite codimension, equal to one or three [1].

In this note we exhibit two kinds of symplectic structures on $\mathrm{Diff}^+(S^1)$. This is done by embedding the group as the coadjoint orbit of a non-regular moment in $\mathrm{Vect}(S^1)^*$ and in the Virasoro space of moments with non-zero central charge.

2. Groups of diffeomorphisms and coadjoint action

Here, the differentiability class is assumed to be C^∞, and the sets of differentiable maps are equiped with the C^∞ topology. The group $\mathrm{Diff}(M)$ of all diffeomorphisms of a compact connected n-manifold M, is a Fréchet Lie group modeled on $\mathrm{Vect}(M)$, the space of differentiable vector fields on M, which plays the role of the Lie algebra for $\mathrm{Diff}(M)$ (see [2]). The adjoint action of $\mathrm{Diff}(M)$ is the natural one on vector fields, namely

$$\mathrm{Ad}(a)(\xi) = Ta \circ \xi \circ a^{-1} \quad \text{for all} \quad a \in \mathrm{Diff}(M),\ \xi \in \mathrm{Vect}(M)$$

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A topological dual of the Lie algebra Vect$(M)^*$, the so-called space of moments, may be identified with the tensor product

$$\mathrm{Vect}(M)^* \approx \Omega^1(M) \underset{C^\infty(M)}{\otimes} D'(M)$$

where $D'(M)$ is the space of distributions on M. The duality is then given by

$$\langle \alpha \otimes T \mid \xi \rangle = \langle T \mid \alpha(\xi) \rangle.$$

The coadjoint action is the contragredient one:

$$\mathrm{Ad}^*(a)(\alpha \otimes T)(\xi) = \langle T \mid \alpha(\mathrm{Ad}(a^{-1})(\xi)) \rangle.$$

One finds in Vect$(M)^*$ two priviliged subspaces, which are everywhere dense for the weak topology, namely

– $\mathfrak{g}_F^*$, defined by the finitely supported distributions. It is established that $\mathfrak{g}_F^*$ is the disjoint union of all finite dimensional orbits (which are all identified) [1].

– $\mathfrak{g}_R^*$, the so-called space of regular moments, defined by the distributions given by n-forms ω. We have, in this case

$$\langle \alpha \otimes \omega \mid \xi \rangle = \int_M \alpha(\xi)\omega.$$

If $M = S^1$, we shall denote a vector field by $\xi = f(\theta)\partial/\partial\theta$, where f is C^∞ and 2π-periodic.

A regular moment μ is then described by a quadratic differential: $\mu = p(\theta)\mathrm{d}\theta^2$. Thus the previous relation becomes

$$\langle \mu \mid \xi \rangle = \int_0^{2\pi} p(\theta) f(\theta)\mathrm{d}\theta.$$

It will be more convenient to describe the circle diffeomorphisms by their universal covering $\mathrm{Diff}^+(\mathbb{R})_{2\pi}$:

$$a \in \mathrm{Diff}^+(\mathbb{R})_{2\pi} \Leftrightarrow a \in \mathrm{Diff}^+(\mathbb{R}) \text{ and } a(x + 2\pi) = a(x) + 2\pi$$

the coadjoint action is then given for any $a \in \mathrm{Diff}(\mathbb{R})_{2\pi}$, by the relation

$$\mathrm{Ad}^*(a)(\mu) = p \circ a^{-1}(a^{-1})'^2 \text{ if } \mu = p(\theta)d\theta^2.$$

The classification of orbits was completed in [1] for all the “Morse moments” (p is transverse to the real axis on a finite subset). The isotropy subgroups may in this case be S^1 itself (modulo a conjugacy) or of finite order. However for certain regular moments (other than Morse) the isotropy subgroup may be trivial. This fact is pointed out but not proved in [4].

3. A non-regular moment

We define a distribution on S^1:

$$T(f) = \sum_{n\in\mathbb{N}} \frac{f(n)}{2^n},$$

where f is 2π-periodic and differentiable. This will define a moment on Diff(S^1); for any vector field $\xi = f(\theta)\partial/\partial\theta$ we have

$$\langle T \otimes \mathrm{d}\theta \mid \xi\rangle = \sum_{n\in\mathbb{N}} \frac{f(n)}{2^n}.$$

The set $e^{i\mathbb{Z}}$ defines a sequence everywhere dense in the circle. This sequence is chosen as an illustrating exemple. Most of the following results would subsist with an other dense sequence.

For convenience we shall calculate on the real line level modulo 2π; this equivalence is now denoted by $[2\pi]$. For every subset A of the real line, we denote by $[A]_{2\pi}$ the set of all elements in $[0, 2\pi]$ which are equivalent to points of A.

The coadjoint action of $a \in \mathrm{Diff}(\mathbb{R})_{2\pi}$ on $T \otimes d\theta$ is then given by

$$\mathrm{Ad}^*(a)(T \otimes \mathrm{d}\theta)(\xi) = \sum_{n\in\mathbb{N}} \frac{f(a(n))}{a'(n)2^n}.$$

Theorem 1. *The isotropy subgroup of $T \otimes \mathrm{d}\theta$, for the coadjoint action of* Diff(S^1), *is reduced to the identity.*

The proof follows from the next three lemmas.

The isotropy subgroup of $T \otimes \mathrm{d}\theta$ is made of diffeomorphisms $\phi \in \mathrm{Diff}(\mathbb{R})_{2\pi}$ such that for every differentiable 2π-periodic function f we have

$$\mathrm{Ad}^*(\phi)(T \otimes \mathrm{d}\theta)\left(f(\theta)\frac{\partial}{\partial\theta}\right) = \sum_{n\in\mathbb{N}} \frac{f(\phi(n))}{\phi'(n)2^n} = \sum_{n\in\mathbb{N}} \frac{f(n)}{2^n}. \qquad (1)$$

Given such a ϕ, we have to proof that ϕ is a translation $x \mapsto x + 2k\pi$, $k \in \mathbb{Z}$. By density argument, it will be sufficient to demonstrate that $[\mathbb{N}]$ is fixed by the action of ϕ. The technique of proof distinguishes the integers n such $n = \phi(m)$ modulo 2π and $m \neq n$. This is the meaning of the set S defined below.

Let S be the set defined by

$$S = \{n \in \mathbb{N} \mid \phi^{-1}(n) \in \mathbb{N} - \{n\} \bmod 2\pi\}$$

The points of S are therefore not $[2\pi]$ fixed by ϕ. The first step is the following

Lemma 1. $\mathbb{N} - S$ *is fixed by ϕ mod 2π.*

Proof. Let $n_0 \notin S$, let us suppose that $\phi(n_0) \neq n_0 \bmod 2\pi$. We define a function f such that $0 \leq f \leq 1$, $f(n_0) = 1$ and f is supported by a little interval around

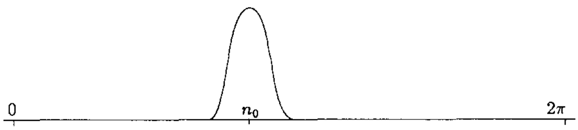

Fig. 1. The function f.

n_0 which doesn't contain $\phi(n_0)$ $[2\pi]$; the picture (see Fig. 1) is then reproduced by periodicity. The relation (1) becomes

$$\sum_{\phi(n)\in \mathrm{Supp} f} \frac{f(\phi(n))}{\phi'(n)2^n} = \frac{1}{2^{n_0}} + \sum_{n\in \mathrm{Supp} f-\{n_0\}} \frac{f(n)}{2^n}; \tag{2}$$

the first sum is done over the integers such that $\phi(n) \in \mathrm{Supp} f$, the second one is over the integers inside $\mathrm{Supp} f$ except n_0. Given an arbitrary rank N, the fact that the integers are all distinct $[2\pi]$ allows a choice for f in sort of $n \in \mathrm{Supp} f \Rightarrow N < n$; thus the second summation in (2) is upperbounded by the partial sum of a convergent series, and so may be arbitrarly small. It follows that

$$\sum_{\phi(n)\in \mathrm{Supp} f} \frac{f(\phi(n))}{\phi'(n)2^n} = \frac{1}{2^{n_0}} + \varepsilon.$$

The hypothesis $\phi(n_0) \neq n_0$ $[2\pi]$ and $n_0 \notin S$, implies that all the integers of the first sum in (2) differ from n_0, it follows that this sum may also be made arbitrarly small; thus a good choice of f allows that (2) becomes

$$\varepsilon' = \frac{1}{2^{n_0}} + \varepsilon,$$

where ε and ε' are arbritrarly small, which is impossible since n_0 is given. The contradiction implies that $\phi(n_0) = n_0$ $[2\pi]$. □

So ϕ has id_{S^1} as a projection, on any interval in which $[\mathbb{N}-S]_{2\pi}$ is everywhere dense.

Lemma 2. *If x is in the closure of $[S]_{2\pi}$ then $(\phi^{-1})'(x) \in 2^{\mathbb{Z}} - \{1\}$.*

Proof. If $n_0 \in S$ then there exists an unique integer, other than n_0, such that $\phi(m_0) = n_0$ $[2\pi]$. The relation which characterizes ϕ becomes

$$\frac{1}{\phi'(m_0)2^{m_0}} + \sum_{\phi(n)\in \mathrm{Supp} f, n\neq m_0} \frac{f(\phi(n))}{\phi'(n)2^n} = \frac{1}{2^{n_0}} + \sum_{n\in \mathrm{Supp} f-\{n_0\}} \frac{f(n)}{2^n}.$$

Here f is chosen as in the proof of lemma 1. Similar arguments will produce a similar conclusion: both expansions in the previous equality may be made arbitrarly small, the equality requires then that $\phi'(m_0) = 2^{n_0-m_0} = 1/[(\phi^{-1})'(n_0)]$ where $n_0 \neq m_0$.

If x is a limit point of a sequence $[n_k]$ of points in $[S]_{2\pi}$, $(\phi^{-1})'(x)$ is then the limit of $(\phi^{-1})'(n_k)$ in $2^{\mathbb{Z}} - \{1\}$. There is no other accumulation point of $2^{\mathbb{Z}}$ than zero, but $(\phi^{-1})'(x) \neq 0$ (ϕ is diffeomorphism); this implies that $(\phi^{-1})'(x) \in 2^{\mathbb{Z}} - \{1\}$. □

Lemma 3. *S is the empty set.*

Proof. Lemma 2 has as a consequence that $(\phi^{-1})' \in 2^{\mathbb{Z}} - \{1\}$ is a constant on any interval in which S is dense $[2\pi]$. This implies that S is not dense mod 2π in $[0, 2\pi]$, while $[0, 2\pi] \ni x \mapsto ax + b$ extends to a diffeomorphism of $\mathrm{Diff}(\mathbb{R})_{2\pi}$ only if $a = 1$. So the complement in $[0, 2\pi]$ of the closure of $[S]_{2\pi}$ is then open and non-empty, and thus is a disjoint union of intervals in which $\mathbb{N} - S$ is $[2\pi]$ dense, hence on those intervals one has $(\phi^{-1})'(x) = 1$ (following lemma 1); but $\mathbb{N} = S \cup (\mathbb{N} - S)$ mod 2π is everywher dense in $[0, 2\pi]$, finally the function $x \mapsto (\phi^{-1})'(x)$ takes necessarly values in $\{1\} \cup (2^{\mathbb{Z}} - \{1\})$, and therefore is a constant equal to one (using one more time the argument of $x \mapsto ax + b$). This requires then the emptyness of S. □

The proof of theorem 1 is then a immediate consequence of lemma 1. Thus the coadjoint orbit of $T \otimes \mathrm{d}\theta$ may be identified to $\mathrm{Diff}(\mathrm{S}^1)$, which inherits the symplectic structure of the orbit.

In view to precise this structure we must recall the fact that a vector field $\xi \in \mathrm{Vect}(M)$ is an element of the tangent space at the neutral element of $\mathrm{Diff}(M)$, and then defines a left infinitesimal action on $\mathrm{Diff}(M)$. One may check that this infinetisemal action is given by $\mathrm{Diff}(M) \ni a \mapsto \xi \circ a$ which is now a vector field on $\mathrm{Diff}(M)$. Thus at point $a \in \mathrm{Diff}(\mathrm{S}^1)$, the two-form on the orbit evaluated on two fundamental vector fields is given by

$$\Omega_a \left(f \circ a(\theta) \frac{\partial}{\partial \theta}\right) \left(g \circ a(\theta) \frac{\partial}{\partial \theta}\right) = \sum_{n \in \mathbb{N}} \frac{(f'g - fg')(a(n))}{a'(n) 2^n}.$$

The definition of "symplectic form" is different here from the finite dimensional case: Ω is a regular, closed two form on $\mathrm{Diff}(\mathrm{S}^1)$, but doesn't define an onto map from the tangent to the cotangent space of the orbit.

4. Central extension and coadjoint action

A central extension of $\mathrm{Diff}(\mathrm{S}^1)$ by $\mathbb{R}$, is given by an exact sequence

$$0 \to \mathbb{R} \to \mathrm{Diff}(\mathrm{S}^1) \rtimes \mathbb{R} \to \mathrm{Diff}(\mathrm{S}^1) \to 0,$$

where $\mathrm{Diff}(\mathrm{S}^1) \rtimes \mathbb{R}$ is the Cartesian product $\mathrm{Diff}(\mathrm{S}^1) \times \mathbb{R}$ equiped with a modified group multiplication, namely: $(\phi, x)(\psi, y) = (\phi \circ \psi, x + y + B(\phi, \psi))$; B satisfying the following group cocycle condition: $B(\phi\psi, \eta) + B(\phi, \psi) =$

$B(\phi, \psi\eta) + B(\psi, \eta)$. In the present case B is the *Bott cocycle* defined by $B(\phi, \psi) = \int_0^{2\pi} \mathrm{Log}(\phi \circ \psi)' \mathrm{d}[\mathrm{Log}(\psi')]$. The diffeomorphisms ϕ and ψ are at the covering level $\mathrm{Diff}(\mathbb{R})_{2\pi}$. This extension which is universal in a sense is called the *Virasoro group* . This group appears in string theory, it is also a symmetry group for Hill's equations. The corresponding Lie algebra is isomorphic to $\mathrm{Vect}(\mathrm{S}^1) \oplus \mathbb{R}$ equipped with a modified bracket [3], namely

$$\left[\left(f(\theta)\frac{\partial}{\partial\theta}, t\right), \left(g(\theta)\frac{\partial}{\partial\theta}, r\right)\right] = \left(\left[f(\theta)\frac{\partial}{\partial\theta}, g(\theta)\frac{\partial}{\partial\theta}\right], \int_0^{2\pi} (f'g'' - f''g')(\theta)\mathrm{d}\theta\right).$$

The space of moments is then made out of couples $(\mu, c) \in \mathrm{Vect}^*(\mathrm{S}^1) \oplus \mathbb{R}$, c is the so called *central charge.* As $\mathbb{R}$ is central, only the $\mathrm{Diff}(\mathrm{S}^1)$ component has a contribution to the coadjoint action. For any diffeomorphism a of the circle we have

$$\mathrm{Ad}^*(a)(\mu, c) = (\mathrm{Ad}^*(a)(\mu) + cw(a), c).$$

$w(a)$ is a quantity which only depends on the extension cocycle. The computation made by differentiation of the inner automorphisms leads to

$$w(a)\left(f(\theta)\frac{\partial}{\partial\theta}\right) = \int_0^{2\pi} S(a) \circ a^{-1}(\theta) f(\theta)\, \mathrm{d}\theta$$

where S is the *Schwarzian derivative* : $S(a) = a'''/a' - \frac{3}{2}(a''/a')^2$.

Using Lie algebra tools, Kirillov has shown that, for a regular moment (p, c) with non-zero central charge, the isotropy subroup is never discrete, *a fortiori* never trivial. On the other hand $(T \otimes \mathrm{d}\theta, c)$, where T is the moment previously defined, shall not admit another fixing diffeomorphism than the identity; indeed such a diffeomorphism must satisfy the following relation:

$$\sum_{n\in\mathbb{N}} \frac{f(\phi(n))}{\phi'(n)2^n} + c\int_0^{2\pi} S(\phi) \circ \phi^{-1}(\theta) f(\theta)\mathrm{d}\theta = \sum_{n\in\mathbb{N}} \frac{f(n)}{2^n}$$

and this for any differentiable 2π-periodic function f. By a suitable choice of f as in the proof of lemma 1, the contribution of the integral will be proportionnal to the length of the support of f in $[0, 2\pi]$ and then may be made arbitrary small. It then suffices to reproduce the proof of theorem 1 to conclude that ϕ covers the identity of S^1. Thus the orbit of $(T \otimes \mathrm{d}\theta, c)$ is an embedding of $\mathrm{Diff}(\mathrm{S}^1)$.

The induced symplectic structure on $\mathrm{Diff}(\mathrm{S}^1)$ at $a \in \mathrm{Diff}(\mathrm{S}^1)$, and evaluated on two fundamentals vector fields has the following expression:

$$\Omega_a\left(f\circ a(\theta)\frac{\partial}{\partial\theta}\right)\left(g\circ a(\theta)\frac{\partial}{\partial\theta}\right)=\sum_{n\in\mathbb{N}}\frac{(f'g-fg')(a(n))}{a'(n)2^n}$$
$$+c\int_0^{2\pi}S(a)\circ a^{-1}(f'g-fg')(\theta)\,\mathrm{d}\theta+c\int_0^{2\pi}(f'g''-g'f'')(\theta)\,\mathrm{d}\theta.$$

5. Questions

Do those structures depend essentially on the choice of the kind of the series $\sum 1/2^n$, or of the choice of the sequence e^{in}? And would it be possible to define representations of Diff(S^1) using those orbits?

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